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DUAL INTEGRAL EQUATIONS

by



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ABSTRACT

The main object of this thesis is to review the work relating to dual integral equations. Most of the methods for solving dual integral equations, involving various special functions as kernels, have been outlined and various extensions of earlier works indicated, in brief.

We also consider triple integral equations and discrete series analogues. In the last chapter we introduce and solve quadruple integral equations by using operators of fractional integration.

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TABLE OF CONTENTS

| | <u>Page</u> |
|--|-------------|
| ABSTRACT | (i) |
| ACKNOWLEDGEMENTS | (ii) |
| CHAPTER I INTRODUCTION | 1 |
| CHAPTER II REVIEW | 7 |
| CHAPTER III A SET OF n -DUAL INTEGRAL EQUATIONS . . . | 47 |
| CHAPTER IV QUADRUPLE INTEGRAL EQUATIONS | 54 |
| BIBLIOGRAPHY | 69 |

CHAPTER I

INTRODUCTION

Dual Integral Equations have their origin in mixed boundary value problems arising in potential theory. Such problems with axial symmetry for an infinite plate involve the solution of Laplace's equation in cylindrical coordinates (ρ, Z) in the region $0 < \rho < \infty$, $0 < Z < t$ under given conditions on the plane boundaries $Z = 0$, $Z = t$. If the boundary conditions are such that either the potential and its normal derivative, or a linear combination of the potential and its derivative are specified over the entire plane face of the plate, an easy way of solving it is by the use of the Hankel transform. The use of this transform eliminates the radial variable and the problem reduces to the solution of a simple differential equation in one independent variable Z with known conditions on $Z = 0$, $Z = t$.

On the other hand, if the condition on the boundary is a mixed one, i.e., if the unknown function satisfies one integral equation over the interval $(0, a)$, and a different equation over the interval (a, ∞) , the solution by a Hankel transform reduces to that of a pair of dual integral equations. For example, let $\rho(r, Z)$ be the potential of a flat circular

electrified disk of conducting material, with centre at the origin, and its axis along the axis of Z . The potential satisfies the partial differential equation

$$(1.1) \quad \frac{\partial^2 \rho}{\partial r^2} + \frac{1}{r} \frac{\partial \rho}{\partial r} + \frac{\partial^2 \rho}{\partial Z^2} = 0$$

Let $\rho^*(s, Z)$ denote the Hankel transform of $\rho(r, Z)$ so that

$$(1.2) \quad \rho^*(s, Z) = \int_0^\infty r \rho(r, Z) J_0(rs) dr, \quad Z > 0$$

Differentiating (1.2) partially twice w.r.t., Z , we get

$$(1.3) \quad \frac{\partial^2 \rho^*}{\partial Z^2} = \int_0^\infty r \frac{\partial^2 \rho}{\partial Z^2} J_0(rs) dr.$$

We substitute the value of $\frac{\partial^2 \rho}{\partial Z^2}$ from equation (1.1) in (1.3) and get

$$(1.4) \quad \frac{\partial^2 \rho^*}{\partial Z^2} = - \int_0^\infty \left(r \frac{\partial^2 \rho}{\partial r^2} + \frac{\partial \rho}{\partial r} \right) J_0(rs) dr.$$

Using the relation

$$\int_0^\infty r \frac{\partial^2 \rho}{\partial r^2} J_0(rs) dr = - \int_0^\infty \frac{\partial \rho}{\partial r} \{J_0(rs) + rs J_0'(rs)\} dr$$

(1.4) becomes

$$\begin{aligned}
 \frac{\partial^2 \rho^*}{\partial Z^2} &= \int_0^\infty \frac{\partial \rho}{\partial r} r s J_0'(rs) dr \\
 &= -s \int_0^\infty \rho \{J_0'(rs) + rs J_0''(rs)\} dr \\
 (1.5) \qquad &= s^2 \int_0^\infty \rho r J_0(rs) dr.
 \end{aligned}$$

We substitute (1.2) in (1.5) and get

$$(1.6) \qquad \frac{\partial^2 \rho^*}{\partial Z^2} = s^2 \rho^*$$

Solving (1.6), we have

$$\rho^* = A(s)e^{-sZ} + B(s)e^{sZ}$$

since $\rho^* \rightarrow 0$, as $Z \rightarrow \infty$ we have clearly $B(s) = 0$

$$(1.7) \qquad \therefore \rho^* = A(s)e^{-sZ}$$

Applying Hankel's theorem to (1.7), we get

$$(1.8) \qquad \rho(r, Z) = \int_0^\infty s A(s) e^{-sZ} J_0(rs) ds.$$

The boundary conditions for a disk of unit radius are

$$\rho = \text{constant} \quad (Z = 0, \quad 0 < r < 1);$$

$$\frac{\partial \rho}{\partial Z} = 0 \quad (Z = 0, \quad r > 1).$$

Substituting $sA(s) = g(s)$, it follows from (1.8) that $g(s)$ should satisfy

$$(1.9a) \quad \int_0^{\infty} g(s) J_0(rs) ds = h(r) \quad 0 < r < 1$$

$$(1.9b) \quad \int_0^{\infty} s g(s) J_0(rs) ds = 0 \quad r > 1$$

In general, the dual integral equations can be written in the form

$$(1.10a) \quad \int_0^{\infty} w(y) g(y) K(y, x) dy = f(x) \quad 0 < x < a$$

$$(1.10b) \quad \int_0^{\infty} g(y) K(y, x) dy = h(x) \quad x > a$$

where $K(y, x)$ is called the kernel function, $w(y)$ the weight function; $f(x)$, $h(x)$ are known function and g is to be

determined.

In this thesis we shall review the general history and development of the theory of dual integral equations in addition to giving some extension of earlier works.

In particular we shall present in Chapter II a historical survey of the theory and methods as well as indicate various extensions of dual integral equations. In Chapter III we find out a solution of the set of n -dual integral equations with Bessel function kernels

$$(1.11a) \quad \int_0^{\infty} D(\alpha)\phi(\alpha)J_{\nu}(\alpha t)d\alpha = f(t) \quad 0 < t < 1$$

$$(1.11b) \quad \int_0^{\infty} E(\alpha)\phi(\alpha)J_{\nu}(\alpha t)d\alpha = g(t) \quad t > 1$$

where $D(\alpha) = \|D_{rs}\|$, $E(\alpha) = \|E_{rs}\|$ are non-singular matrices of known functions; $r, s = 1, 2, \dots, n$; $f(t)$, $g(t)$ and $\phi(\alpha)$ are columns with n coordinates. The method used is similar to Szefer's method, in which case Szefer [77] has solved this problem with $g(t) \equiv 0$. A solution for the set of n -dual integral equations with trigonometric kernels can also be obtained by applying the same technique. In Chapter IV, we define quadruple integral equations by

$$(1.12a) \quad \int_0^{\infty} \xi^{-2\alpha} \psi(\xi) J_{\nu}(\rho\xi) d\xi = F_1(\rho) \quad 0 < \rho < a$$

$$(1.12b) \quad \int_0^{\infty} \xi^{-2\beta} \psi(\xi) J_{\nu}(\rho\xi) d\xi = G_2(\rho) \quad a < \rho < b$$

$$(1.12c) \quad \int_0^{\infty} \xi^{-2\alpha} \psi(\xi) J_{\nu}(\rho\xi) d\xi = F_3(\rho) \quad b < \rho < c$$

$$(1.12d) \quad \int_0^{\infty} \xi^{-2\beta} \psi(\xi) J_{\nu}(\rho\xi) d\xi = G_4(\rho) \quad \rho > c$$

and solve them by using fractional integrals. The solution is obtained in terms of Fredholm's integral equation of the second kind.

The analysis used, throughout this thesis, is formal and no attempt is made to justify the integration, differentiation or the change of the order of integration.

Finally we give a fairly extensive bibliography on the subject.

CHAPTER II

REVIEW

Most of dual integral equations with which we meet in the solution of mixed boundary value problems are of the type

$$(2.1a) \quad \int_0^{\infty} w(x)A(x)K(x,y)dx = f(y) \quad y \in I_1$$

$$(2.1b) \quad \int_0^{\infty} A(x)K(x,y)dx = g(y) \quad y \in I_2$$

where $w(x)$ is a weight function; $f(y)$, $g(y)$ are known functions defined on I_1 and I_2 respectively where $I_1 = (0,a)$, $I_2 = (a,\infty)$; $K(x,y)$ is a kernel function defined over the whole xy -plane and $A(x)$ is a function to be determined.

It seems that Weber [91] was the first to consider dual integral equations. He formulated equations of this type and derived (by inspection) the solution for the case in which $w(x) = x^{-1}$, $K(x,y) = J_0(xy)$, $f(y) \equiv 1$ and $g(y) \equiv 0$. Berltramí [4] gave a direct solution for the same values of $w(x)$, $K(x,y)$ and $g(y)$ but by taking $f(y)$ arbitrary. No work in this field appears to have been done after that for over half a century, until Titchmarsh [81] initiated the first systematic treatment for the

solution of dual integral equations. Applying Parseval's formula for Mellin transforms,

$$(2.2) \quad \int_0^{\infty} f(x)g(x)dx = \frac{1}{2\pi i} \int_{K-i\infty}^{K+i\infty} F(s)G(1-s)ds$$

where $F(s)$ and $G(s)$ are the Mellin transforms of $f(x)$ and $g(x)$ respectively, to the left hand sides of the equations

$$(2.3a) \quad \int_0^{\infty} f(u)J_0(\rho u)du = g(\rho) \quad 0 < \rho < 1$$

$$(2.3b) \quad \int_0^{\infty} f(u)uJ_0(\rho u)du = 0 \quad \rho > 1$$

the value of $F(s)$ is obtained, from which $f(x)$ can be determined by Mellin's inversion formula. Titchmarsh [81] then goes on to solve the general pair of dual integral equations

$$(2.4a) \quad \int_0^{\infty} x^{\alpha}A(x)J_{\nu}(xy)dx = f(y) \quad 0 < y < 1$$

$$(2.4b) \quad \int_0^{\infty} A(x)J_{\nu}(xy)dx = g(y) \quad y > 1$$

with $\alpha > 0$ and $g(y) \equiv 0$. Busbridge [8] considered conditions for the validity of Titchmarsh solution and proved that if

$0 < \alpha < 2$ or $-2 < \alpha < 0$ and $-\nu - 1 < \alpha - \frac{1}{2} < \nu + 1$, then the equations (2.4) have one and only one solution, $A(x)$, in the class of functions whose Mellin transforms are regular for $-\nu - 1 < \sigma < \alpha$ and $O(|t|^{\sigma-\alpha+\varepsilon})$, for every $\varepsilon > 0$, in any interior strip.

Copson [19] suggested that if one starts with a more suitable form of potential function, a single integral equation is obtained involving a repeated integral, and this can be solved by two applications of the known solution of Abel's integral equation.

Tranter [83] considered the dual integral equations (2.4) with $\alpha = \pm 1$, $g(y) \neq 0$. By applying Hankel inversion formula to (2.4b) and substituting the value of $A(x)$ so obtained in equation (2.4a), the solution was obtained in the form of a Schlomilch's integral equation

$$(2.5) \quad \int_0^{\rho} s^{2\nu+1} (y^2 - s^2)^{-\frac{1}{2}} \chi(s) ds = \left(\frac{1}{2} \pi\right)^{\frac{1}{2}} y^{\nu+1} P(y)$$

where

$$(2.6) \quad A(x) = H(x) + \left(\frac{1}{2} \pi x\right)^{\frac{1}{2}} \int_0^1 s^{\nu+\frac{1}{2}} \chi(s) J_{\nu+\frac{1}{2}}(xs) ds$$

and

$$(2.7) \qquad H(x) = g(1)J_{\nu+1}(x) + x \int_1^{\infty} yg(y)J_{\nu}(xy)dy$$

Gordon [36] solved the same pair of dual integral equations for general values of $\alpha > -1$. Assuming $A(x) = A_1(x) + A_2(x)$ and choosing $A_2(x)$ such that

$$(2.8) \qquad \int_0^{\infty} A_2(x)J_{\nu}(xy)dx = \begin{cases} 0 & 0 < y < 1 \\ g(y) & y > 1 \end{cases}$$

to get

$$(2.9a) \qquad \int_0^{\infty} x^{\alpha}A_1(x)J_{\nu}(xy)dx = f(y) - \int_0^{\infty} x^{\alpha}A_2(x)J_{\nu}(xy)dx \qquad 0 < y < 1$$

$$(2.9b) \qquad \int_0^{\infty} A_1(x)J_{\nu}(xy)dx = 0 \qquad y > 1.$$

$A_2(x)$ is determined from (2.8) and $A_1(x)$ from the pair (2.9), hence $A(x)$ is found.

An alternative approach is to modify the two integrands in such a way that they do in fact become the same. For this purpose, the following Sonine's integral [90]

$$(2.10) \qquad J_{\mu+\nu+1}(Z) = \frac{Z^{\mu+1}}{2^{\mu}\Gamma(\mu+1)} \int_0^1 J_{\nu}(\rho Z)\rho^{\nu+1}(1-\rho^2)^{\mu}d\rho$$

and the discontinuous integral [90]

$$(2.11) \quad \int_0^{\infty} J_{\lambda}(at) J_{\mu}(bt) t^{1+\mu-\lambda} dt = \begin{cases} 0 & 0 < a < b \\ \frac{b^{\mu}(a^2-b^2)^{\lambda-\mu-1}}{2^{\lambda-\mu-1} a^{\lambda} \Gamma(\lambda-\mu)} & 0 < b < a \end{cases}$$

where $\lambda > \mu > -1$, are used. Noble [50] considered a more general case with $g(y)$ not necessarily equal to zero and $-2 < \alpha < 0$.

We take $f(y) = 0$ and introduce a function $G(x)$ such that

$$(2.12) \quad \int_0^{\infty} x G(x) J_{\nu}(yx) dx = \begin{cases} g(y) & y > 1 \\ 0 & 0 < y < 1. \end{cases}$$

By Hankel's inversion formula, we have

$$(2.13) \quad G(x) = \int_1^{\infty} \xi g(\xi) J_{\nu}(\xi x) d\xi.$$

The integral equations (2.4) with $f(y) = 0$ become

$$(2.14a) \quad \begin{aligned} \int_0^{\infty} x^{\alpha} \{A(x) - xG(x)\} J_{\nu}(yx) dx \\ = - \int_0^{\infty} \lambda^{1+\alpha} G(\lambda) J_{\nu}(y\lambda) d\lambda \quad 0 < y < 1 \end{aligned}$$

$$(2.14b) \quad \int_0^{\infty} \{A(x) - xG(x)\} J_{\nu}(yx) dx = 0 \quad y > 1.$$

The dual integral equations (2.14) can be solved by known methods. The complete solution is obtained by adding this solution to the one determined for the case $g(y) = 0$.

A direct method which is an extension of that given by Copson [19] was considered by Noble [51]. The use is made of the known solution of Abel's integral equation. Let us set the integral equation (2.4b) equal to an unknown function $\chi(y)$ for $(0 < y < 1)$. Inverting by Hankel's theorem, we get

$$(2.15) \quad A(x) = x \int_0^1 \lambda \chi(\lambda) J_\nu(x\lambda) d\lambda + x \int_1^\infty \lambda g(\lambda) J_\nu(x\lambda) d\lambda.$$

Substituting (2.15) in (2.4a), interchanging the order of integration, we obtain

$$(2.16) \quad \chi(\lambda) = - \frac{2\lambda^{\nu-1}}{\Gamma(1+\frac{1}{2}\alpha)\Gamma(-\frac{1}{2}\alpha)} \frac{d}{d\lambda} \int_\lambda^1 s(s^2-\lambda^2)^{\frac{1}{2}\alpha} G(s) ds$$

if $-2 < \alpha < 0$ and where

$$(2.17) \quad G(s) = \frac{\Gamma(-\frac{1}{2}\alpha)}{\Gamma(1+\frac{1}{2}\alpha)} \frac{s^{-2\nu-\alpha-1}}{2^{1+\alpha}} \frac{d}{ds} \int_0^s y^{\nu+1} (s^2-y^2)^{\frac{1}{2}\alpha} f(y) dy \\ - \int_1^\infty \lambda^{1-\nu} g(\lambda) (\lambda^2-s^2)^{-1-\frac{1}{2}\alpha} d\lambda \quad (0 < s < 1)$$

The case for $0 < \alpha < 2$ is treated similarly.

Sneddon [59] gave an elementary solution of the integral equations (2.4) in the case $\alpha = \pm 1$, $\nu = 0$, $g(y) \equiv 0$. For $\alpha = -1$, set

$$(2.18) \quad A(x) = x \int_0^1 \phi(t) \cos x t dt.$$

Substituting this value of $A(x)$ in (2.4b), the integral equation is automatically satisfied, if we use the integral [90]

$$(2.19) \quad \int_0^\infty J_0(\xi \rho) \sin(\xi t) d\xi = \begin{cases} 0 & t < \rho \\ (t^2 - \rho^2)^{-1/2} & \rho < t \end{cases}$$

Now substitute (2.18) in (2.4a), then making use of another well known integral [90]

$$(2.20) \quad \int_0^\infty J_0(\rho \xi) \cos(\xi t) d\xi = \begin{cases} 0 & \rho < t \\ (\rho^2 - t^2)^{-1/2} & t < \rho \end{cases}$$

we obtain the value of $\phi(t)$ in terms of Schlömilch's integral equation.

Copson [20] gave an elegant solution in the general case with $\nu \neq 0$, $g(y) = 0$ and $0 < \alpha < 2$, $-2 < \alpha < 0$, by putting

$$(2.21) \quad A(x) = x^{1-\frac{\alpha}{2}} \int_0^1 \phi(t) J_{\nu+\frac{\alpha}{2}}(xt) dt.$$

In the course of treatment, solution of Abel's integral equation and the integral (2.11) have been used.

Lowengrub and Sneddon [46] extended Copson's method [20] directly. They solved the problem with $f(y) \equiv 0$, $g(y) \neq 0$ and added this to the Copson's solution of the pair (2.4) with $g(y) \equiv 0$. To this end, we set

$$(2.22) \quad A(x) = x^{1-\frac{\alpha}{2}} \int_1^\infty \phi(t) J_{\nu+\alpha}(xt) dt.$$

Using (2.11) we see that the equation (2.4a) is satisfied by this value of $A(x)$ and substituting (2.22) in (a.4b), we get the value of $\phi(t)$, so that $A(x)$ is determined.

We observe that in the methods given by Titchmarsh [81], Tranter [83], Copson [20], and Noble [51], the use of integral transforms in solving dual integral equations is quite complicated and laborious. The analysis is involved and long. Erdelyi and Sneddon [30] introduced the operational approach for solution. The fractional integration operators $I_{\eta,\alpha}, K_{\eta,\alpha}$ derived by Erdelyi and Kober [29] and the modified operator $S_{\eta,\alpha}$ of Hankel transforms were used. They are defined by

$$(2.23) \quad I_{\eta, \alpha} f(x) = \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} y^{\eta-\alpha} f(y) dy \quad \alpha > 0, \eta > -\frac{1}{2}$$

$$(2.24) \quad K_{\eta, \alpha} f(x) = \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^{\infty} (y-x)^{\alpha-1} y^{-\eta-\alpha} f(y) dy \quad \alpha > 0, \eta > -\frac{1}{2}$$

$$(2.25) \quad S_{\eta, \alpha} f(x) = x^{-\frac{1}{2}\alpha} \int_0^{\infty} y^{-\frac{1}{2}\alpha} J_{2\eta+\alpha}(2\sqrt{xy}) f(y) dy$$

so that

$$(2.26) \quad I_{\eta+\alpha, \beta} S_{\eta, \alpha} = S_{\eta, \alpha+\beta}$$

$$(2.27) \quad K_{\eta, \alpha} S_{\eta+\alpha, \beta} = S_{\eta, \alpha+\beta}.$$

Taking $\alpha = 2\beta$ in the integral equations (2.4), we obtain

$$(2.28) \quad I_{\frac{1}{2}\nu+\beta, -\beta} f = I_{\frac{1}{2}\nu+\beta, -\beta} S_{\frac{1}{2}\nu-\beta, 2\beta} A = S_{\frac{1}{2}\nu-\beta, \beta} A$$

$$(2.29) \quad K_{\frac{1}{2}\nu-\beta, \beta} g = K_{\frac{1}{2}\nu-\beta, \beta} S_{\frac{1}{2}\nu, 0} A = S_{\frac{1}{2}\nu-\beta, \beta} A$$

If we now define a function h by

$$(2.30) \quad h = \begin{cases} I_{\frac{1}{2}\nu+\beta, -\beta} f & \text{on } (0, 1) \\ K_{\frac{1}{2}\nu-\beta, \beta} g & \text{on } (1, \infty) \end{cases}$$

then h can be calculated and hence $A(x)$ is determined.

For the case, when $\alpha = -1$, $\nu = 0$, $f(y) \equiv 0$, in the integral equations (2.4), Lowengrub and Sneddon [45] found an elementary method by setting

$$(2.31) \quad A(x) = x \int_1^{\infty} \phi(t) \cos(xt) dt$$

where $\phi(t)$ is such that the integral converges, this implies

$$(2.32) \quad \lim_{t \rightarrow \infty} \phi(t) = 0.$$

Using the solution of Abel's integral equation and applying the discontinuous integrals (2.19), (2.20), the value of $\phi(t)$ is obtained as

$$(2.33) \quad \phi(t) = \frac{2}{\pi} \frac{d}{dt} \int_t^{\infty} \frac{yg(y)dy}{(y^2 - t^2)^{1/2}}.$$

Jovin [39] gave conditions under which the integrands on the left hand side of equations (2.4) become the same and are reduced to

$$(2.34) \quad \int_0^{\infty} x^{\alpha/2} A(x) J_{\nu+\alpha/2}(yx) dx = \begin{cases} g_1(y) & 0 < y < 1 \\ g_2(y) & y > 1 \end{cases}$$

Multiplying factor method was used by Noble [53] to

solve the pair of equations (2.4). We multiply both sides of the integral equation (2.4a) by

$$(2.35) \quad \frac{t^{-\xi-\nu-1}}{2^\xi \Gamma(\xi+1)} y^{\nu+1} (t^2 - y^2)^\xi$$

and use Sonine's integral (2.10). For equation (2.4b) we multiply both sides by

$$(2.36) \quad \frac{t^{\nu-\eta-1}}{2^\eta \Gamma(\eta+1)} y^{-\nu+1} (y^2 - t^2)^\eta$$

and use another formula of Sonine [90]

$$(2.37) \quad t^{-\eta-1} J_{\nu-\eta-1}(tx) = \frac{t^{\nu-\eta-1}}{2^\eta \Gamma(\eta+1)} \int_0^\infty J_\nu[t(s^2+t^2)^{\frac{1}{2}}] (s^2+t^2)^{-\frac{1}{2}\nu} s^{2\eta+1} ds.$$

Integrating the resulting equations and inverting the orders of integration, we solve the two equations by taking the values of ξ, η in such a way that we have the same integrands on the left hand side. In the special case for $f(y) = 0$, Burlak [6] solved the dual integral equations (2.4) by transforming the two equations so as to involve the same integrands. Hence by Hankel's inversion theorem, $A(x)$ can be determined. Recently, Srivastava [72] has solved the integral equations (2.4) for the

case when α is a positive integer greater than unity. The use is made of Sonine's first integral (2.10) and Weber-Schafheitlin integral

$$(2.38) \quad \int_0^{\infty} J_{\nu+2n+1}(x) J_{\nu}(xr) dx = \begin{cases} \frac{(-1)^n r^{\nu}}{n+1} P_n^{(0,\nu)}(2r^2-1) & 0 < r < 1 \\ 0 & r > 1 \end{cases}$$

where $\nu > -1$ and n is a positive integer. The solution is obtained in the form of an integral equation involving Jacobi polynomial as kernel.

William [94] reduced the dual integral equations (2.4) to the form

$$(2.39a) \quad F(x) \equiv \int_0^{\infty} \xi^{\alpha} \psi(\xi) J_n(x\xi) d\xi = f(x) \quad 0 < x < 1$$

$$(2.39b) \quad G(x) \equiv \int_0^{\infty} \psi(\xi) J_n(x\xi) d\xi = x^{-\alpha} g(x) \quad x > 1$$

where $0 < \alpha < 2$ or $-2 < \alpha < 0$.

We denote the Mellin transforms of $F(x)$, $G(x)$ by $F^*(s)$, $G^*(s)$ respectively, then from the convolution theorem of Mellin transforms, we have

$$(2.40) \quad F^*(s) = J_n^*(s) \psi^*(1-s+\alpha)$$

$$(2.41) \quad G^*(s) = J_n^*(s-\alpha)\psi^*(1-s+\alpha).$$

Making repeated use of Mellin transforms and the Hankel's inversion theorem, $\psi(\xi)$ is determined.

Uptil now, we have mostly concentrated on the solution of dual integral equations (2.4). After this, we will consider dual integral equations in which the weight function is of more general nature. Tranter [84] considered the following pair of dual integral equations

$$(2.42a) \quad \int_0^\infty G(u)f(u)J_\nu(\rho u)du = g(\rho) \quad 0 < \rho < 1$$

$$(2.42b) \quad \int_0^\infty f(u)J_\nu(\rho u)du = 0 \quad \rho > 1$$

Note that in (2.4), $G(u) = u^\alpha$.

Let us assume that $f(u)$ has a Neumann series representation

$$(2.43) \quad f(u) = u^{1-K} \sum_0^\infty a_m J_{2m+K+\nu}(u).$$

Substituting (2.43) in (2.42b), we see from

$$(2.44) \quad \int_0^\infty u^{1-K} J_{2m+K+\nu}(u) J_\nu(\rho u) du = 0 \quad (\rho > 1)$$

that the equation is satisfied. On substituting (2.43) in

(2.42a), the coefficients a_m are determined, after a long calculation, in the form of an infinite series of linear equations.

Cooke [14] found an integral which is an analogue of Tranter's solution [84] given by

$$(2.45) \quad f(u) = \frac{2^K}{\Gamma(1-K)} u^{1+K} \int_0^1 h(t) t^{\alpha+1} J_{\nu-1}(ut) dt$$

where $0 < \operatorname{Re}(K) < 1$, and $h(t)$ is a function to be determined. Substituting (2.45) in the pair of equations (2.42), the second equation is satisfied, whereas the first one gives the value of $h(t)$ in the form of a Fredholm's integral equation of the second kind.

Burlak [5] considered the dual integral equations of the form

$$(2.46a) \quad \int_K^\infty u^{-\mu-\nu} (u^2 - K^2)^\alpha \psi(u) J_\mu(xu) du = f(x) \quad 0 \leq x \leq 1$$

$$(2.46b) \quad \int_0^\infty \psi(u) J_\nu(xu) du = g(x) \quad x > 1.$$

Since the equations are linear, we may write

$$A(x) = A_1(x) + A_2(x)$$

where $A_1(x)$ denotes the solution in the case $g \equiv 0$, $A_2(x)$ the solution in the case $f \equiv 0$. The essence of the method is the reduction of the dual integral equations to a single integral equation. The use is made of Sonine's second integral [90]

(2.47)

$$\int_0^\infty J_\mu(bt) \frac{J_\nu\{a(t^2+z^2)^{\frac{1}{2}}\}}{(t^2+z^2)^{\frac{1}{2}\nu}} t^{\mu+1} dt$$

$$= \begin{cases} 0 & 0 < a < b \\ \frac{b^\mu}{a^\nu} \left\{ \frac{(a^2-b^2)^{\frac{1}{2}}}{z} \right\}^{\nu-\mu-1} J_{\nu-\mu-1}\{Z(a^2-b^2)^{\frac{1}{2}}\} & 0 < b < a \end{cases}$$

where $\text{Re}(\nu) > \text{Re}(\mu) > -1$, and of the known solutions of the integral equations

(2.48)

$$\int_a^x \psi(\rho) (x^2-\rho^2)^{\frac{1}{2}\beta} J_\beta\{K(x^2-\rho^2)^{\frac{1}{2}}\} d\rho = h(x)$$

and

(2.49)

$$\int_x^\infty \psi(\rho) (\rho^2-x^2)^{\frac{1}{2}\beta} I_\beta\{K(\rho^2-x^2)^{\frac{1}{2}}\} d\rho = m(x).$$

Dwivedi [23] used the multiplying factor method to solve the equations

(2.50a)

$$\int_K^\infty u^{-\mu-K} J_\mu(xu) \{1+H(u)\} \psi(u) du = f(x) \quad 0 \leq x < d$$

$$(2.50b) \quad \int_0^{\infty} (u^2 - K^2)^{\alpha} J_{\nu}(xu) \psi(u) du = g(x) \quad x > d.$$

We multiply both sides of (2.50a) by

$$(2.51) \quad J_p \{s(\rho^2 - x^2)^{\frac{1}{2}}\} (\rho^2 - x^2)^{p/2} x^{\mu+1},$$

integrate both sides from 0 to ρ and apply Sonine's integral [90]. On substituting $s = iK$ and $\sqrt{u^2 - K^2} = \xi$, we have

$$(2.52) \quad \int_0^{\infty} \xi^{-\mu-p} \{\sqrt{(\xi^2 + K^2)}\}^{-\nu+1} J_{\mu+p+1}(\rho\xi) [1 + H\{\sqrt{(\xi^2 + K^2)}\}] \psi\{\sqrt{(\xi^2 + K^2)}\} d\xi \\ = \rho^{-\mu-p-1} \int_0^{\rho} f(x) x^{\mu+1} J_p \{K\sqrt{(\rho^2 - x^2)}\} \left(\frac{\sqrt{(\rho^2 - x^2)}}{K}\right)^p dx \quad 0 \leq \rho < d$$

provided $\mu > -1$, $p > -1$.

Similarly, we multiply both sides of (2.50b) by

$$(2.53) \quad J_q \{K\sqrt{(x^2 - \rho^2)}\} \{\sqrt{(x^2 - \rho^2)}\}^q x^{-\nu+1}$$

integrate from ρ to ∞ and apply Sonine's integral (2.47). On substituting $\sqrt{u^2 - K^2} = \xi$, we get

$$(2.54) \quad \int_0^{\infty} \xi^{2\alpha-\nu-q} \{\sqrt{(\xi^2 + K^2)}\}^{-\nu-1} J_{\nu-q-1}(\rho\xi) \psi\{\sqrt{(\xi^2 + K^2)}\} d\xi \\ = \rho^{\nu-q-1} \int_{\rho}^{\infty} g(x) x^{-\nu+1} J_q \{K\sqrt{(x^2 - \rho^2)}\} \left(\frac{\sqrt{(x^2 - \rho^2)}}{K}\right)^q dx \quad \rho > d$$

provided $\nu > q > -1$.

The values of p and q are so chosen that the left hand sides of (2.52) and (2.54) become the same. The solution is then obtained by applying Hankel's inversion formula.

Some authors have solved dual integral equations with trigonometric kernels which, in general, are given by

$$(2.55a) \quad \int_0^{\infty} \xi^{\alpha} \psi(\xi) \frac{\sin}{\cos} (x\xi) d\xi = f(x) \quad 0 < x < a$$

$$(2.55b) \quad \int_0^{\infty} \psi(\xi) \frac{\sin}{\cos} (x\xi) d\xi = g(x) \quad x > a.$$

Sneddon [61] considered the system of equations (2.55) with $\alpha = -1$, by transforming the kernel functions into Bessel functions of order one-half. The resulting dual integral equations bear a formal resemblance to the equations solved by Busbridge [8].

However, the restriction $-\nu - 1 < \alpha - \frac{1}{2} < 1 + \nu$ is violated and the use of this solution is not justified. This fact was pointed out by Fredricks [35] who considered the equations (2.55) with $g(x) \equiv 0$. By applying Weber discontinuous integrals (2.19) and (2.20), the solution is determined only in the case where the principle of superposition holds.

Sneddon [60] considered the equations (2.55) with $g(x) \equiv 0$ and derived the solution of the given pair of equations by using a procedure similar to Fredricks. Here we make use of

the Weber discontinuous integrals (2.19), (2.20) and the orthogonal property of the Jacobi polynomials. The solution is given by

$$(2.56) \quad \psi(\xi) = P_0 \xi J_1(\xi) + 2 \sum_{q=1}^{\infty} q p_q J_{2q}(\xi)$$

provided that the constants p_q are so chosen that

$$(2.57) \quad f(x) = p_0 + \sum_{q=1}^{\infty} p_q P_q^{(0, \frac{1}{2})}(x^2).$$

Following the method used by Sneddon [57], Srivastava [70] considered the dual integral equations

$$(2.58a) \quad \int_0^{\infty} y^{2K-1} g(y) \cos(xy) dy = f(x) \quad (0 \leq x < 1)$$

$$(2.58b) \quad \int_0^{\infty} g(y) \cos(xy) dy = 0 \quad (x > 1).$$

In this case $f(x)$ in (2.57) is represented by the Jacobi series

$$(2.59) \quad f(x) = \sum_{n=1}^{\infty} a_n P_n^{(K, \frac{1}{2})}(x^2)$$

and satisfies the condition

$$(2.60) \quad \int_0^1 (1-x^2)^{K-\frac{1}{2}} f(x) dx = 0.$$

Tranter [88] solved the dual integral equations (2.55) by making use of the well known Bessel function integral representations.

$$(2.61) \quad \frac{\pi}{2} J_0(r\xi) = \int_0^r \frac{\cos(\xi x)}{(r^2-x^2)^{\frac{1}{2}}} dx = \int_r^\infty \frac{\sin(\xi x) dx}{(x^2-r^2)^{\frac{1}{2}}}$$

For cosine function, we integrate (2.55b) with respect to 'x', multiply the resulting equation by $(x^2-r^2)^{-\frac{1}{2}}$ and integrate between r, ∞ . The first equation is multiplied by $(r^2-x^2)^{-\frac{1}{2}}$ and integrated from 0 to r , then we get

$$(2.62) \quad \frac{\pi}{2} \int_0^\infty \xi^{-1} \psi(\xi) J_0(r\xi) d\xi = \begin{cases} \int_0^r \frac{f(x) dx}{(r^2-x^2)^{\frac{1}{2}}} & (0 < r < 1) \\ 0 & r > 1 \end{cases}$$

Application of Hankel's inversion theorem then gives the value of $\psi(\xi)$.

In the case of sine kernel, we first differentiate (2.55a) and then proceed in the same way.

Srivastava [71] considered the general pair of dual integral equations given by

$$(2.63a) \quad \int_0^{\infty} t^{2n} g(t) \cos(rt) dt = f(r) \quad 0 < r < 1$$

$$(2.63b) \quad \int_0^{\infty} g(t) \cos(rt) dt = F(r) \quad r > 1$$

where n is a positive integer. Applying Fourier inversion theorem and the results from Ta Li[79], we get, under certain assumptions about the nature of $f(r)$ and $F(r)$, the value of $g(t)$ as

$$(2.64) \quad t^{2n} g(t) = H(t) + (-1)^n \frac{\pi}{2} \int_0^1 s^{-2n} d\{s^{2n+1} q(s)\} J_{2n}(st)$$

where

$$(2.65) \quad H(t) = \frac{\pi}{2} [t^{2n-1} \sin tF(1) + t^{2n} \int_1^{\infty} F(r) \cos(rt) dr],$$

$$(2.66) \quad q(s) = \int_s^1 \frac{P(r) T_{2n+1}(r/s)}{(r^2 - s^2)^{\frac{1}{2}}} dr,$$

$$(2.67) \quad P(r) = \frac{4}{\pi} \left[\int_0^{\infty} H(t) \cos(rt) dt - f(r) \right].$$

Dwivedi [21] solved all the four possible cases of (2.63) with sine and cosine functions by making use of the integral representations for Bessel functions, given by

$$(2.68) \quad J_\nu(ur) = \frac{2^{1-\nu}}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2})} \left(\frac{u}{r}\right)^\nu \int_0^r \frac{\cos(ux)}{(r^2-x^2)^{\frac{1}{2}-\nu}} dx, \quad \operatorname{Re}(\nu) > -\frac{1}{2}$$

$$(2.69) \quad = \frac{2^{1+\nu}}{\Gamma(\frac{1}{2}-\nu)\Gamma(\frac{1}{2})} \left(\frac{r}{u}\right)^\nu \int_r^\infty \frac{\sin(ux)}{(x^2-r^2)^{\frac{1}{2}+\nu}} dx, \quad \operatorname{Re}(\nu) > \frac{1}{2}.$$

Szefer [76] took the more general case of the dual integral equations

$$(2.70a) \quad \int_0^\infty G(\alpha)\phi(\alpha) \frac{\sin(\alpha x)}{\cos(\alpha x)} d\alpha = f(x) \quad 0 < x < 1$$

$$(2.70b) \quad \int_0^\infty \phi(\alpha) \frac{\sin(\alpha x)}{\cos(\alpha x)} d\alpha = 0 \quad x > 1.$$

Taking a trial solution as

$$(2.71) \quad \phi(\alpha) = \int_0^1 g(\xi) J_0(\xi\alpha) d\xi$$

we see that (2.70b) is satisfied. Substituting (2.71) in (2.70a) and putting

$$G(\alpha) = \alpha[1+H(\alpha)]$$

we obtain, on using Weber-Schafheitlin integral, the solution in the form of a Fredholm's integral equation of the second kind.

Srivastav [68] considered the dual integral equations involving inverse Mellin transforms. They are given by

$$(2.72a) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \psi(s) \tan \alpha s \rho^{-s} ds = f_1(\rho) \quad 0 < \rho < 1$$

$$(2.72b) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s\psi(s) \rho^{-s} ds = -f_2(\rho) \quad \rho > 1$$

$$(2.73a) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s\psi(s) \rho^{-s} ds = -f_1(\rho) \quad 0 < \rho < 1$$

$$(2.73b) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \psi(s) \tan \alpha s \rho^{-s} ds = f_2(\rho) \quad \rho > 1$$

$$(2.74a) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \psi(s) \rho^{-s} ds = f_1(\rho) \quad 0 < \rho < 1$$

$$(2.74b) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s\psi(s) \tan \alpha s \rho^{-s} ds = f_2(\rho) \quad \rho > 1$$

$$(2.75a) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s\psi(s) \tan \alpha s \rho^{-s} ds = f_1(\rho) \quad 0 < \rho < 1$$

$$(2.75b) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \psi(s) \rho^{-s} ds = f_2(\rho) \quad \rho > 1$$

For the pair (2.72) with $f_2(\rho) \equiv 0$, we assume that for $0 < \rho < 1$

$$(2.76) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s\psi(s)\rho^{-s}ds = -\rho \frac{\partial}{\partial \rho} \int_0^1 \frac{g_1(t)}{\sqrt{(t^2-\rho^2)}} dt$$

then from the inversion theorem for Mellin transforms, we get

$$(2.77) \quad s\psi(s) = \frac{s\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2})}{2\Gamma(\frac{1}{2}s+\frac{1}{2})} \int_0^1 g_1(t)t^{s-1}dt \quad \operatorname{Re}(s) > -1$$

Rewrite the equation (2.72a) as

$$(2.78) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \psi(s) \tan \frac{1}{2} \pi s \rho^{-s} ds = F_1(\rho) \quad 0 < \rho < 1$$

where

$$(2.79) \quad F_1(\rho) = f_1(\rho) + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \psi(s) (\tan \frac{1}{2} \pi s - \tan \alpha s) \rho^{-s} ds.$$

Solving (2.78) and (2.79), we get the value of $g_1(t)$ in the form of a Fredholm's integral equation of the second kind given by

$$(2.80) \quad g_1(t) = -\frac{2}{\pi} \frac{d}{dt} \int_0^t \frac{\rho f_1(\rho) d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} - \int_0^1 \frac{g(u)}{u} K_1(u,t) du$$

$$(0 < t < 1)$$

where

$$(2.81) \quad K_1(u, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{u}{t}\right)^s (\tan \frac{1}{2} \pi s - \tan \alpha s) \cot \frac{1}{2} \pi s ds.$$

Similarly, the pair (2.72) with $f_1(\rho) \equiv 0$ is solved and the two solutions are added up. Other pairs can be solved in the same way.

Srivastav and Parihar [69] gave a closed form solution for the dual integral equations, involving inverse Mellin transform. The following formulas from Mellin's inversion theorem and integral relations [28] are used. For $c > 0$

$$(2.82) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} h^{-1} B\left(\frac{1}{2}, \frac{s}{h}\right) x^{-s} ds = \begin{cases} (1-x^h)^{-\frac{1}{2}} & \text{for } 0 < x < 1 \\ 0 & \text{for } x > 1, \end{cases}$$

and that, for $c < h/2$

$$(2.83) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} h^{-1} B\left(\frac{1}{2}, \frac{1}{2} - \frac{s}{h}\right) x^{-s} dx = \begin{cases} 0 & \text{for } 0 < x < 1 \\ (x^h - 1)^{-\frac{1}{2}} & \text{for } x > 1. \end{cases}$$

Set $h = \pi/\alpha$. We assume that $f_2(\rho) \equiv 0$ in equation (2.72b)

and that for $0 < \rho < 1$

$$(2.84) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s \psi(s) \rho^{-s} ds = -\rho \frac{\partial}{\partial \rho} \int_{\rho}^1 g_1(t) (t^h - \rho^h)^{-\frac{1}{2}} dt.$$

Using inversion theorem for Mellin transforms and formula (2.82), we get

$$(2.85) \quad \psi(s) = h^{-1} B\left(\frac{1}{2}, \frac{s}{h}\right) \int_0^1 t^{s-h/2} g_1(t) dt,$$

leading to the relation

$$(2.86) \quad \psi(s) \tan \alpha s = h^{-1} B\left(\frac{1}{2}, \frac{1}{2} - \frac{s}{h}\right) \int_0^1 t^{s-h/2} g_1(t) dt,$$

Substitution of (2.86) in (2.72a) gives the value of $g_1(t)$.

Erdélyi [25] solved equations of type (2.72) by using fractional integration operators. The problem was characterised in this way. Suppose that there are two functions f_1 and f_2 only partially known, $f_1(x)$ being given for $0 < x < 1$ and $f_2(x)$ for $x > 1$; whose Mellin transforms satisfy the relation

$$(2.87) \quad \frac{g_1(s)}{g_2(s)} = \frac{\Gamma(1+\alpha-s/m)\Gamma(\xi+s/n)}{\Gamma(1+\alpha-s/m)\Gamma(\xi+\beta+s/n)}$$

with α, β, \dots, n known, then f_1, f_2, g_1, g_2 can be determined.

Erdogan and Bahar [32] considered the following system of simultaneous dual integral equations

$$(2.88a) \quad \int_0^\infty \sum_{j=1}^n a_{1j}(x) f_j(x) J_{\nu_1}(xy) dx = h_1(y) \quad 0 < y < 1$$

$$(2.88b) \quad \int_0^{\infty} \sum_{j=1}^n b_{ij}(x) f_j(x) J_{\mu_i}(xy) dx = g_i(y) \quad y > 1$$

$$i = 1, 2, \dots, n.$$

By substituting

$$(2.89) \quad \psi_i(x) = \sum_{j=1}^n b_{ij}(x) f_j(x) - x \int_1^{\infty} y g_i(y) J_{\mu_i}(xy) dy,$$

$$(2.90) \quad (c_{ij}(x)) = C = AB^{-1}, \quad A = (a_{ij}), \quad B = (b_{ij}),$$

$$(2.91) \quad p_i(y) = h_i(y) - \int_0^{\infty} \sum_{j=1}^n c_{ij}(x) J_{\mu_i}(xy) x dx \int_1^{\infty} t g_i(t) J_{\mu_j}(xt) dt,$$

the system (2.88) is reduced to

$$(2.92a) \quad \int_0^{\infty} \sum_{j=1}^n c_{ij}(x) \psi_j(x) J_{\mu_i}(xy) dx = p_i(y) \quad 0 < y < 1$$

$$(2.92b) \quad \int_0^{\infty} \psi_i(x) J_{\mu_i}(xy) dx = 0 \quad y > 1;$$

$$i = 1, 2, \dots, n.$$

We set

$$(2.93) \quad \psi_j(x) = x^{1-\beta_j} \sum_{m=0}^{\infty} A_{jm} J_{\mu_j+2m+\beta_j}(x), \quad j = 1, 2, \dots, n.$$

By this choice of $\psi_j(x)$, (2.92b) is automatically satisfied and substitution of (2.93) in (2.92a) gives the value of coefficients A_{jm} .

Westmann [92] considered the following pairs of dual integral equations

$$(2.94a) \quad \int_0^{\infty} [a\psi_1(\xi) + \psi_2(\xi)] J_{\nu+2}(\xi r) d\xi = f_1(r) \quad 1 < r < \infty$$

$$(2.94b) \quad \int_0^{\infty} [b\psi_1(\xi) + \psi_2(\xi)] \xi^{2\alpha} J_{\nu+2}(\xi r) d\xi = f_2(r) \quad 0 < r < 1$$

and

$$(2.95a) \quad \int_0^{\infty} [c\psi_1(\xi) + \psi_2(\xi)] J_{\nu}(\xi r) d\xi = f_3(r) \quad 1 < r < \infty$$

$$(2.95b) \quad \int_0^{\infty} [\psi_1(\xi) + \psi_2(\xi)] \xi^{2\alpha} J_{\nu}(\xi r) d\xi = f_4(r) \quad 0 < r < 1.$$

Following Copson [20] and Lowengrub and Sneddon [43], the separate cases are considered by taking $f_i(r) \equiv 0$ except one of them and then assuming the representation for $\psi_1(\xi), \psi_2(\xi)$ in terms of unknown functions, which are determined by actual substitution of the assumed values of $\psi_1(\xi)$ and $\psi_2(\xi)$ in the system of equations (2.94), (2.95).

Szefer [77] considered the set of n -dual integral equations given by

$$(2.96a) \quad \int_0^{\infty} D(\alpha) \phi(\alpha) J_p(\alpha t) d\alpha = f(t) \quad 0 < t < 1$$

$$(2.96b) \quad \int_0^{\infty} E(\alpha) \phi(\alpha) J_p(\alpha t) d\alpha = 0 \quad t > 1$$

where $D(\alpha)$, $E(\alpha)$ are $n \times n$ matrices, $|E(\alpha)| \neq 0$; $\phi(\alpha)$, $f(t)$ are column matrices of order n . We put $E(\alpha)\phi(\alpha) = \psi(\alpha)$, so that

$$\phi(\alpha) = E^{-1}(\alpha)\psi(\alpha) \quad \because |E(\alpha)| \neq 0$$

$$\therefore D(\alpha)\phi(\alpha) = D(\alpha)E^{-1}(\alpha)\psi(\alpha) = F(\alpha)\psi(\alpha)$$

and equations (2.96) become

$$(2.97a) \quad \int_0^{\infty} F(\alpha)\psi(\alpha) J_p(\alpha t) dt = f(t) \quad 0 < t < 1$$

$$(2.97b) \quad \int_0^{\infty} \psi(\alpha) J_p(\alpha t) dt = 0 \quad t > 1.$$

Again, we set

$$(2.98) \quad \psi(\alpha) = \int_0^1 \xi^{p+1} g(\xi) J_{p+1}(\alpha \xi) d\xi.$$

On substituting (2.98) in (2.97b), it is automatically satisfied in view of the integral (2.11) and substitution in (2.97a) gives the value of $g(\xi)$ in terms of a set of Fredholm's integral equations of the second kind, which can be reduced to a single Fredholm's equation of that kind.

Erdogan [31] considered the following system of simultaneous integral equations:

$$(2.99) \quad \frac{2}{\pi} \int_0^{\infty} P(\xi) \cos y\xi d\xi = p_0(y) \quad (y \in L')$$

$$\frac{2}{\pi} \int_0^{\infty} Q(\xi) \sin y\xi d\xi = q_0(y) \quad (y \in L')$$

$$(2.100) \quad \lim_{x \rightarrow +0} \frac{1}{\pi} \int_0^{\infty} [(a_{11} + a_{12}x\xi)P(\xi) - (a_{13} + a_{14}x\xi)Q(\xi)]e^{-x\xi} \sin y\xi d\xi = g_1(y) \\ (y \in L)$$

$$\lim_{x \rightarrow +0} \frac{1}{\pi} \int_0^{\infty} [(a_{21} + a_{22}x\xi)P(\xi) - (a_{23} + a_{24}x\xi)Q(\xi)]e^{-x\xi} \cos x\xi d\xi = g_2(y) \\ (y \in L)$$

where L is the union of nonintersecting finite segments

$L_K (L = \Sigma L_K)$, on $0 \leq y < \infty$ and L' is the complement of L . Taking into account the symmetry conditions and setting

$$(2.101) \quad P(\xi) = \int_0^{\infty} p(t) \cos \xi t dt, \quad Q(\xi) = \int_0^{\infty} q(t) \sin \xi t dt$$

the solution is obtained in the form of simultaneous singular integral equations.

Keer [41] considered the equations (2.94), (2.95).

Assuming trial solutions analogous to Westmann, the problem was solved by making use of the operators of fractional integration.

Another extension was given by Fox [34]. He considered dual integral equations of much more general nature involving H-function kernels, given by

$$(2.102a) \quad \int_0^{\infty} H(ux \mid \begin{smallmatrix} \alpha_1, a_1 \\ \beta_1, a_1 \end{smallmatrix}; n) f(u) du = g(x) \quad 0 < x < 1$$

$$(2.102b) \quad \int_0^{\infty} H(ux \mid \begin{smallmatrix} \lambda_1, a_1 \\ \mu_1, a_1 \end{smallmatrix}; n) f(u) du = h(x) \quad x > 1$$

where

$$(2.103) \quad H(x \mid \begin{smallmatrix} \alpha_1, a_1 \\ \beta_1, a_1 \end{smallmatrix}; n) = H(x \mid \begin{smallmatrix} \alpha_1, a_1 & \alpha_2, a_2 & \dots & \alpha_n, a_n \\ \beta_1, a_1 & \beta_2, a_2 & \dots & \beta_n, a_n \end{smallmatrix})$$

$$= \frac{1}{2\pi i} \int_c \prod_{i=1}^n \left\{ \frac{\Gamma(\alpha_i + sa_i)}{\Gamma(\beta_i - sa_i)} \right\} x^{-s} ds.$$

The path c is a straight line $x = \sigma$ and σ is such that $\sigma > -\alpha_i/a_i$, $2\sigma \sum_{i=1}^n a_i < \sum_{i=1}^n (\beta_i - \alpha_i)$. We apply Parseval's

formula for Mellin transforms (2.2) to the pair of integral

equations (2.102) and transform the denominator of H-function in the first equation to that of the second and the numerator of the second to that of the first. Finally, the inverse Mellin transform gives the value of $f(u)$. Saxena [57] generalized Fox's result and using the same technique obtained a solution of the following dual integral equations

$$(2.104a) \quad \int_0^{\infty} H_{m,2p+m}^{p,m}(xu | (\gamma_i, \alpha_i), (1-\delta_i, \alpha_i), (1-\beta_k, t_k))^{(1-\alpha_k, t_k)} f(u) du = \zeta_1(x)$$

$$0 < x < 1$$

$$(2.104b) \quad \int_0^{\infty} H_{n,2p+n}^{p+n,0}(xu | (\lambda_\ell, \xi_\ell), (\nu_i, \alpha_i), (1-\rho_i, \alpha_i))^{(\mu_\ell, \lambda_\ell)} f(u) du = \zeta_2(x)$$

$$x > 1$$

$$\text{where } H_{n,m}^{p,q} = \frac{1}{2\pi i} \int_C \frac{\prod_{i=1}^p \Gamma(\beta_i + B_i s) \prod_{i=1}^q (1-\alpha_i - A_i s) x^{-s}}{\prod_{i=p+1}^m \Gamma(1-\beta_i - B_i s) \prod_{i=q+1}^n \Gamma(\alpha_i + A_i s)} ds$$

the contour C being a straight line $x = \sigma$ and σ is such that (i) $\sigma < -\beta_i/B_i$, (ii) $\sigma < (1-\alpha_i)/A_i$, (iii) σ fulfills the conditions of convergence ([27], section (1.19)).

Buschman [9] pointed out that if we define

$$(2.105) \quad I^{n,\alpha,A}(x) = \frac{A}{\Gamma(\alpha)} (x^A - 1)^{\alpha-1} x^{A\eta - A\alpha} H(x-1)$$

where $H(x)$ is the Heaviside step function, then the corresponding convolution theorem for Erdelyi-Kober operators of fractional integration is

$$(2.106) \qquad I_{x^A}^{\eta,\alpha} f(x) = (I^{\eta,\alpha,A} * f)(x)$$

where $I^{\eta,\alpha,A}(x) \in L'(0,\infty)$, $f \in L'(0,\infty)$, $\alpha > 0$, $\eta > \frac{1}{A} - 1$.

We have similar theorem for $K_{x^A}^{\eta,\alpha}(x)$. Kesarwani [42] used this idea to solve dual integral equations involving Meijer's function. By application of Mellin transforms and fractional integration operators, the parameters in the kernels of the two equations are made equal and then the inverse Mellin transform gives the solution. As the dual integral equations (2.102) and (2.104) can be reduced to dual convolution transform after exponential change of variables, Tanno [80] considered the following equations

$$(2.107a) \qquad \int_{-\infty}^{\infty} G(x-t)\phi(t)dt = f(x) \qquad x > \lambda$$

$$(2.107b) \qquad \int_{-\infty}^{\infty} H(x-t)\phi(t)dt = g(x) \qquad x < \lambda$$

where $G(t)$ and $H(t)$ are generated by certain meromorphic functions and λ is a constant. In order to transform the equations (2.107) with a common kernel, we use the uniqueness

theorem of bilateral transform given by

$$(2.108) \quad G_* G^*(x) = H_* H^*(x), \quad \text{for all } x,$$

where $G_* G^*$ is the dual convolution transform of G . This procedure is justifiable only under certain conditions.

Some authors have considered triple integral equations of the form

$$(2.109a) \quad \int_0^\infty \phi(u) J_\nu(ru) du = f(r) \quad 0 < r < a$$

$$(2.109b) \quad \int_0^\infty u^{2p} \phi(u) J_\nu(ru) du = g(r) \quad a < r < b$$

$$(2.109c) \quad \int_0^\infty \phi(u) J_\nu(ru) du = h(r) \quad r > b$$

where $\phi(u)$ is unknown. Tranter [86] solved this problem for the case $h(r) \equiv 0$ and obtained the value of $\phi(u)$ in terms of a series. Setting

$$(2.110) \quad \phi(u) = u^{-p} \sum_{n=1}^{\infty} C_n J_{\nu+2n-1+p}(bu)$$

we see that the equation (2.109c) is automatically satisfied

by this value. Substitution of (2.110) in the first two equations leads to dual series and the determination of the coefficients C_n from these series completes the solution. The convergence of the integrals occurring in calculation requires that

$$\operatorname{Re}(\nu) > -1 \quad \text{when} \quad p = \frac{1}{2}, \quad \operatorname{Re}(\nu) > -\frac{1}{2} \quad \text{when} \quad p = -\frac{1}{2}.$$

Cooke [15] made use of the fact that the equations are linear and substituted

$$(2.111) \quad \begin{aligned} \int_0^{\infty} u^{2p} \phi(u) J_{\nu}(ru) du &= f_1(r) & 0 < r < a \\ &= f_2(r) & r > b. \end{aligned}$$

In this way, we get two pairs of dual integral equations

$$(2.112) \quad \left\{ \begin{aligned} \int_0^{\infty} u^{2p} \phi(u) J_{\nu}(ru) du &= f_1(r) & 0 < r < a \\ &= g(r) & a < r < b \\ \int_0^{\infty} \phi(u) J_{\nu}(ru) du &= h(r) & r > b \end{aligned} \right.$$

and

$$(2.113) \quad \left\{ \begin{aligned} \int_0^{\infty} \phi(u) J_{\nu}(ru) du &= f(r) & 0 < r < a \\ \int_0^{\infty} u^{2p} \phi(u) J_{\nu}(ru) du &= g(r) & a < r < b \\ &= f_2(r) & r > b \end{aligned} \right.$$

From (2.112) and (2.113), we find $f_1(r)$ and $f_2(r)$ by applying Noble's method [53]. So $\phi(u)$ can be determined by Hankel's inversion formula. The equations (2.109) can also be solved by putting

$$\phi(u) = \phi_1(u) + \phi_2(u)$$

and writing

$$(2.114) \quad \left\{ \begin{array}{ll} \int_0^\infty u^{2p} \phi_1(u) J_\nu(ru) du = g_1(r) & 0 < r < b \\ \int_0^\infty u^{2p} \phi_2(u) J_\nu(ru) du = g_2(r) & a < r < b. \end{array} \right.$$

With these substitutions, we again get two simultaneous pairs of integral equations, which can be solved by known methods.

Cooke [16] solved the integral equations (2.109) by transforming them into the form in which $f(r) \equiv 0$ and $h(r) \equiv 0$. This is always possible, since by taking

$$\begin{array}{ll} (2.115a) & f(r) \quad 0 < r < a \\ (2.115b) & \int_0^\infty \phi(u) J_\nu(ru) du = \ell(r) \quad a < r < b \\ (2.115c) & h(r) \quad b < r < \infty \end{array}$$

$\Phi(u)$ can be determined by applying Hankel inversion theorem and so we can find a function $m(r)$, which is such that

$$(2.116) \quad \int_0^{\infty} u^{2p} \Phi(u) J_{\nu}(ru) du = m(r).$$

Write $\psi(u) = \phi(u) - \Phi(u)$ and subtract equations (2.115a), (2.116) and (2.115c) from equations (2.109a), (2.109b) and (2.109c) respectively, then we have

$$(2.117a) \quad \int_0^{\infty} \psi(u) J_{\nu}(ru) du = 0 \quad 0 < r < a$$

$$(2.117b) \quad \int_0^{\infty} u^{2p} \psi(u) J_{\nu}(ru) du = g(r) - m(r) \quad a < r < b$$

$$(2.117c) \quad \int_0^{\infty} \psi(u) J_{\nu}(ru) du = 0 \quad r > b$$

Now to solve the equations (2.117), we assume that

$$(2.118) \quad \int_0^{\infty} \psi(u) J_{\nu}(ru) du = \chi(r) \quad a < r < b$$

Substitution of (2.118) in (2.117b) gives on inverting the order of integration and simplification a Fredholm's integral

equation of the first kind.

Still another method for solving the integral equations (2.109) was given by Cooke [17]. The problem is solved by making use of Erdelyi-Kober operators $I_{\eta,\alpha}$, $K_{\eta,\alpha}$ and the modified operator $S_{\eta,\alpha}$ of Hankel transforms.

The solutions obtained by various authors Williams [97] and Cooke [16] are reobtained by making use of the operational approach and the results agree with those already got.

Dwivedi [22] considered the more general triple integral equations given by

$$(2.119a) \quad \int_0^{\infty} \psi(u) J_{\mu}(xu) du = f(x) \quad 0 < x < a$$

$$(2.119b) \quad \int_0^{\infty} u^{\alpha} \psi(u) J_{\nu}(xu) du = g(x) \quad a < x < b$$

$$(2.119c) \quad \int_0^{\infty} \psi(u) J_{\lambda}(xu) du = h(x) \quad b < x < \infty$$

Following Cooke [15], we substitute $\psi(u) = \psi_1(u) + \psi_2(u)$ where $\psi_1(u)$ and $\psi_2(u)$ are determined by the equations (2.119a), (2.119c) and

$$(2.120a) \quad \int_0^{\infty} u^{\alpha} \psi_1(u) J_{\nu}(xu) du = g_1(x) \quad (0 < x < b)$$

$$(2.120b) \quad \int_0^{\infty} u^{\alpha} \psi_2(u) J_{\nu}(xu) du = g_2(x) \quad (a < x < \infty)$$

This gives us two pairs of dual integral equations which can be solved by the analysis used by Noble [53]. Hence the problem is solved.

The discrete analogues of dual and triple integral equations are dual and triple series equations. Such equations are obtained, if we make use of series solution of Laplace's equation. Dual relations involving Fourier-Bessel series, Dini series, trigonometric series, series of Jacobi polynomials and that of Laguerre polynomials have been solved by various authors. For example, a pair of dual equations involving series of Jacobi polynomials is given by

$$(2.121a) \quad \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+n+1)\Gamma(\beta+n+3/2)} P_n^{(\alpha,\beta)}(\cos \theta) = F(\theta) \quad 0 \leq \theta < \phi$$

$$(2.121b) \quad \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\beta+n+1)\Gamma(\alpha+n+1/2)} P_n^{(\alpha,\beta)}(\cos \theta) = G(\theta) \quad \phi < \theta \leq \pi$$

where $\alpha > -\frac{1}{2}$, $\beta > -1$ and $P_n^{(\alpha,\beta)}(\cos \theta)$ denotes the Jacobi

polynomial. Srivastav [66] considered this problem in two stages. In the first case, we put $G(\theta) \equiv 0$ and assume that, when $0 \leq \theta < \phi$

$$(2.122) \quad \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\beta+n+1)\Gamma(\alpha+n+1/2)} P_n^{(\alpha,\beta)}(\cos \theta) \\ = -(\cos \frac{1}{2}\theta)^{-2\beta} \operatorname{cosec} \theta \frac{d}{d\theta} \int_{\theta}^{\phi} \frac{h_1(u) du}{\sqrt{(\cos \theta - \cos u)}}.$$

Using orthogonality relation for the Jacobi polynomials, we get on simplification

$$(2.123) \quad A_n = \frac{\sqrt{\pi} q_n(\alpha,\beta)}{2\sqrt{2}} \int_0^{\phi} h_1(u) \sin(\frac{1}{2}u)^{2\alpha-1} P_n^{(\alpha-\frac{1}{2},\beta+\frac{1}{2})}(\cos u) du$$

where $q_n(\alpha,\beta) = (\alpha+\beta+2n+1)n!\Gamma(\alpha+\beta+n+1)$.

Substituting the value of A_n from (2.123) in (2.121a) and solving for $h_1(u)$, we obtain

$$h_1(u) = \frac{2}{\pi} (\sin \frac{1}{2}u)^{1-2\alpha} (\cos \frac{1}{2}u)^{1+2\beta} \frac{d}{du} \int_0^u \frac{F(\theta) (\sin \frac{1}{2}\theta)^{2\alpha} \sin \theta d\theta}{\sqrt{(\cos \theta - \cos u)}}.$$

In the second case, we put $F(\theta) \equiv 0$ and determine the value of A_n in the same way. The solution of the general problem is now obtained by merely adding the two solutions.

Recently, Askey [3] has considered a type of discrete dual equations which he calls dual sequence equations. A typical example is

$$(2.124a) \quad \int_{-1}^1 (1-x)^c f(x) P_n^{(\alpha, \beta)}(x) (1-x)^\alpha (1+x)^\beta dx = a_n \quad n = 0, 1, \dots, N$$

$$(2.124b) \quad \int_{-1}^1 f(x) P_n^{(\alpha, \beta)}(x) (1-x)^\alpha (1+x)^\beta dx = b_n \quad n = N+1, \dots$$

The author mentioned that there are a number of results, given by Feldheim [33], Al-Salam [2] and others, for the orthogonal polynomials which contain Sonine's formulas (2.10) and (2.47) as limiting cases. Some of these formulas can be used to play the same role for dual series and dual sequence equations that we have seen the Sonine integrals to play for dual integral equations. The solution of dual sequence equations involving Laguerre and Jacobi polynomials has been obtained under certain conditions.

CHAPTER III

A SET OF n-DUAL INTEGRAL EQUATIONS

In this chapter we find a solution for a given set of n-dual integral equations. Making use of matrix notation and applying the discontinuous integral

$$(3.1) \quad \int_0^{\infty} J_{\nu}(\alpha t) J_{\nu-1}(\alpha x) d\alpha = \begin{cases} 0 & t < x \\ x^{\nu-1}/t^{\nu} & t > x \end{cases}$$

provided $\operatorname{Re}(\nu) > 0$,

the solution is obtained in terms of a Fredholm's integral equation of the second type. We shall employ a method similar to Szefer [76]. It is primarily based on the method used by Copson [20] and later on extended by Lowengrub and Sneddon [46].

1. Transformation into canonical form. Let us consider the set of n-dual integral equations with Bessel function kernels.

$$(3.2a) \quad \int_0^{\infty} D(\alpha) \phi(\alpha) J_{\nu}(\alpha t) d\alpha = f(t) \quad 0 < t < 1$$

$$(3.2b) \quad \int_0^{\infty} E(\alpha) \phi(\alpha) J_{\nu}(\alpha t) d\alpha = g(t) \quad t > 1$$

where $D(\alpha) = \|D_{rs}\|$, $E(\alpha) = \|E_{rs}\|$ are non-singular matrices of known functions; $r, s = 1, 2, \dots, n$; $f(t)$, $g(t)$ are column vectors whose elements are known functions (with n coordinates); $\phi(\alpha)$ -- a column vector whose elements are unknown functions (with n coordinates) and $J_\nu(\alpha t)$ is the Bessel function of the first kind and of order ν . Szefer [77] has solved the problem with $g(t) \equiv 0$. Here we shall consider the set of equations

$$(3.3a) \quad \int_0^\infty D(\alpha)\phi(\alpha)J_\nu(\alpha t)d\alpha = 0 \quad 0 < t < 1$$

$$(3.3b) \quad \int_0^\infty E(\alpha)\phi(\alpha)J_\nu(\alpha t)d\alpha = g(t) \quad t > 1$$

since once the solution of this set of equations is known, the solution of the set (3.2) can be obtained merely by adding Szefer's solution to it.

In order to solve the set of equations (3.3), we shall transform them into a convenient form by substituting

$$(3.4) \quad D(\alpha)\phi(\alpha) = \chi(\alpha).$$

The relations (3.4) give us a column vector of unknown functions (with n coordinates) and since $D(\alpha)$ is non-singular so that it has an inverse. Thus we can write (3.4) in

the form

$$(3.5) \quad \phi(\alpha) = D^{-1}(\alpha)\chi(\alpha).$$

Equations (3.3), can be rewritten as

$$(3.6a) \quad \int_0^{\infty} \chi(\alpha) J_{\nu}(\alpha t) d\alpha = 0 \quad 0 < t < 1$$

$$(3.6b) \quad \int_0^{\infty} L(\alpha)\chi(\alpha) J_{\nu}(\alpha t) d\alpha = g(t) \quad t > 1$$

where the functional matrix $L(\alpha)$ is given by

$$L(\alpha) = E(\alpha)D^{-1}(\alpha).$$

The set of equations (3.6) will be called the set of dual integral equations in the canonical form and it is this set whose solution will be sought.

2. Solution of the equations (3.6). We seek the solution of the set of equations (3.6) in the form

$$(3.7) \quad \chi(\alpha) = t^{\nu} \int_1^{\infty} \ell(x) J_{\nu-1}(\alpha x) dx ,$$

where $\ell(x)$ is an unknown vector. Substituting the value of

$\chi(\alpha)$ from (3.7) in (3.6a), we get

$$\int_0^{\infty} t^{\nu} \int_1^{\infty} \ell(x) J_{\nu-1}(\alpha x) J_{\nu}(\alpha t) dx d\alpha \quad 0 < t < 1.$$

Inverting the order of integration, it becomes

$$(3.8) \quad \int_1^{\infty} t^{\nu} \ell(x) dx \int_0^{\infty} J_{\nu}(\alpha t) J_{\nu-1}(\alpha x) d\alpha \quad 0 < t < 1.$$

Since $x > 1$, we observe that $t < x$ always; hence applying the discontinuous integral (3.1), the integral (3.8) vanishes identically and consequently, the equations (3.6a) are satisfied.

Now we substitute (3.7) in (3.6b), then we have

$$(3.9) \quad \int_0^{\infty} L(\alpha) [t^{\nu} \int_1^{\infty} \ell(x) J_{\nu-1}(\alpha x) dx] J_{\nu}(\alpha t) d\alpha = g(t) \quad t > 1.$$

Assuming for the matrix $L(\alpha)$

$$(3.10) \quad L(\alpha) = I + V(\alpha)$$

where I is the unit matrix and

$$(3.11) \qquad V(\alpha) = \begin{pmatrix} V_{11}(\alpha), L_{12}(\alpha), \dots, L_{1n}(\alpha) \\ \dots\dots\dots \\ \dots\dots\dots \\ \dots\dots\dots \\ L_{n1}(\alpha), L_{n2}(\alpha), \dots, V_{nn}(\alpha) \end{pmatrix} ; \quad V_{ss}(\alpha) = L_{ss}(\alpha) - I$$

we obtain from (3.9) that

$$(3.12) \qquad g(t) = \int_0^\infty I[t^\nu \int_1^\infty \ell(x) J_\nu(\alpha t) J_{\nu-1}(\alpha x) dx] d\alpha \\ + \int_0^\infty V(\alpha) [t^\nu \int_1^\infty \ell(x) J_\nu(\alpha t) J_{\nu-1}(\alpha x) dx] d\alpha \; .$$

Inverting the order of integration in (3.12), we have

$$g(t) = \int_1^\infty t^\nu \ell(x) dx \int_0^\infty J_\nu(\alpha t) J_{\nu-1}(\alpha x) d\alpha \\ + \int_1^\infty [\int_0^\infty V(\alpha) J_\nu(\alpha t) J_{\nu-1}(\alpha x) d\alpha] t^\nu \ell(x) dx \; .$$

Again applying (3.1), we obtain

$$(3.13) \qquad g(t) = \int_1^t x^{\nu-1} \ell(x) dx + t^\nu \int_1^\infty [\int_0^\infty V(\alpha) J_\nu(\alpha t) J_{\nu-1}(\alpha x) d\alpha] \ell(x) dx \; .$$

Putting

$$(3.14) \quad \int_1^t x^{\nu-1} \ell(x) dx = Q(t)$$

and assuming that

$$(3.15) \quad \lim_{t \rightarrow \infty} Q(t) = 0$$

we have

$$(3.16) \quad t^{\nu-1} \ell(t) = Q'(t) .$$

Substituting (3.14) and (3.16) in (3.13), we obtain

$$(3.17) \quad Q(t) + t^{\nu} \int_1^{\infty} \left[\int_0^{\infty} V(\alpha) J_{\nu}(\alpha t) J_{\nu-1}(\alpha x) d\alpha \right] x^{1-\nu} Q'(x) dx = g(t) .$$

Putting

$$(3.18) \quad \int_0^{\infty} V(\alpha) J_{\nu}(\alpha t) J_{\nu-1}(\alpha x) d\alpha = N(x, t) ,$$

equations (3.17) may be written as

$$(3.19) \quad Q(t) + t^{\nu} \int_1^{\infty} N(x, t) x^{1-\nu} Q'(x) dx = g(t) .$$

Integrating by parts and making use of the condition (3.15), we get

$$(3.20) \quad Q(t) - t^{\nu} \int_1^{\infty} [N(x,t)x^{1-\nu}]' Q(x) dx = g(t).$$

Taking

$$(3.21) \quad [N(x,t)x^{1-\nu}]' = -K(x,t),$$

we finally obtain

$$(3.22) \quad Q(t) + t^{\nu} \int_1^{\infty} K(x,t)Q(x)dx = g(t).$$

This is a set of Fredholm's integral equations of the second kind, which can be reduced to a single equation of that type.

Hence the problem is formally solved.

Note that for $n = 1$, the problem reduces to Tranter's case [84].

CHAPTER IV

QUADRUPLE INTEGRAL EQUATIONS

In the analysis of mixed boundary value problems in which different conditions hold over two different parts of the same boundary can often be easily reduced to the solution of dual integral equations. Such equations involving Bessel or trigonometric functions as kernels have now amassed a considerable literature. In the current decade Tranter [87], Cooke [15], Dwivedi [22] and a few other authors have considered the problems with different conditions over three different parts of the same boundary. The integral equations involved in the solution of such problems were termed as triple integral equations. In some boundary value problems, however, the boundary condition is such that different conditions hold over four different parts of the boundary and, in such cases, the integral equations involved can be written in the symmetrical form

$$(4.1a) \quad L_1(\alpha, \rho) \equiv \int_0^{\infty} \xi^{-2\alpha} \psi(\xi) J_\nu(\rho \xi) d\xi = F_1(\rho) \quad 0 < \rho < a$$

$$(4.1b) \quad L_2(\beta, \rho) \equiv \int_0^{\infty} \xi^{-2\beta} \psi(\xi) J_\nu(\rho \xi) d\xi = G_2(\rho) \quad a < \rho < b$$

$$(4.1c) \quad L_3(\alpha, \rho) \equiv \int_0^\infty \xi^{-2\alpha} \psi(\xi) J_\nu(\rho \xi) d\xi = F_3(\rho) \quad b < \rho < c$$

$$(4.1d) \quad L_4(\beta, \rho) \equiv \int_0^\infty \xi^{-2\beta} \psi(\xi) J_\nu(\rho \xi) d\xi = G_4(\rho) \quad \rho > c$$

where F_1 , G_2 , F_3 , and G_4 are known functions of ρ and $\psi(\xi)$ is to be determined. We shall call such equations as 'Quadruple Integral Equations'. In this chapter we derive a solution of the integral equations (4.1), in the case when $G_2 \equiv 0 \equiv G_4$, by using operators of fractional integration and those of Hankel transforms. We will show that this can be extended to the case in which G_2 and G_4 are not necessarily zero.

1. Operators. We give here a brief account of the definitions and properties of the operators used in solving the integral equations (4.1). We define operators ${}_a^b I_{\eta, \alpha}$ and ${}_c^d K_{\eta, \alpha}$ by the formulae

$$(4.2) \quad {}_a^b I_{\eta, \alpha} f(x) = \frac{x^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_a^b (x^2 - u^2)^{\alpha-1} u^{2\eta+1} f(u) du \quad \alpha > 0$$

$$(4.3) \quad {}_a^b I_{\eta, \alpha} f(x) = \frac{x^{-2\eta-2\alpha-1}}{\Gamma(1+\alpha)} \frac{d}{dx} \int_a^b (x^2 - u^2)^\alpha u^{2\eta+1} f(u) du \quad -1 < \alpha < 0$$

$$(4.4) \quad {}_c^d K_{\eta, \alpha} f(x) = \frac{2x^{2\eta}}{\Gamma(\alpha)} \int_c^d (u^2 - x^2)^{\alpha-1} u^{-2\alpha-2\eta+1} f(u) du \quad \alpha > 0$$

$$(4.5) \quad {}_c^d K_{\eta, \alpha} f(x) = -\frac{x^{2\eta-1}}{\Gamma(1+\alpha)} \frac{d}{dx} \int_c^d (u^2 - x^2)^{\alpha} u^{-2\alpha-2\eta+1} f(u) du \quad -1 < \alpha < 0$$

For $\alpha = 0$, these are just identity operators. Note that with these definitions ${}_0^x I_{\eta, \alpha}$ and ${}_x^\infty K_{\eta, \alpha}$ are simply the Erdelyi-Kober operators [61]. In these cases we will drop the indices on the left and write them as $I_{\eta, \alpha}$ and $K_{\eta, \alpha}$. We also observe that (4.2), (4.3) make sense if $b < x$ and similarly (4.4), (4.5) are defined only if $c > x$.

The modified operator $S_{\eta, \alpha}$ of the Hankel transforms is defined by

$$(4.6) \quad S_{\eta, \alpha} f(x) = 2^\alpha x^{-\alpha} \int_0^\infty t^{1-\alpha} J_{2\eta+\alpha}(xt) f(t) dt.$$

Sneddon [61] has shown the following relations between the Erdelyi-Kober and Hankel operators

$$(4.7) \quad I_{\eta+\alpha, \beta} S_{\eta, \alpha} = S_{\eta, \alpha+\beta}$$

$$(4.8) \quad K_{\eta, \alpha} S_{\eta+\alpha, \beta} = S_{\eta, \alpha+\beta}$$

$$(4.9) \quad S_{n+\alpha, \beta} S_{n, \alpha} = I_{n, \alpha+\beta}$$

$$(4.10) \quad S_{n, \alpha} S_{n, \alpha+\beta} = K_{n, \alpha+\beta}$$

provided the conditions for the existence of the various operators are satisfied. The inverse operators are

$$(4.11) \quad \frac{b_I^{-1}}{a_{n, \alpha}} = \frac{b_I}{a_{n+\alpha, -\alpha}}$$

$$(4.12) \quad \frac{d_K^{-1}}{c_{n, \alpha}} = \frac{d_K}{c_{n+\alpha, -\alpha}}$$

$$(4.13) \quad S_{n, \alpha}^{-1} = S_{n+\alpha, -\alpha}$$

We give two lemmas, which define the product of pairs of operators.

Lemma 4.1. Let $\frac{f_I}{e_{n, \alpha}}, \frac{x_I^{-1}}{d_{n, \alpha}}$ be operators as given in (4.2),
 (4.3) and (4.11). Then

$$(4.14) \quad \frac{x_I^{-1}}{d_{n, \alpha}} \frac{f_I}{e_{n, \alpha}} f(x) = \frac{2 \sin \pi \alpha}{\pi} x^{-2n} (x^2 - d^2)^{-\alpha} \int_e^f \frac{(d^2 - t^2)^\alpha t^{2n+1} f(t)}{x^2 - t^2} dt$$

provided $x > d \geq f > e$.

Proof: We shall prove it in two steps. First, for
 $0 < \alpha < 1$ and then for $-1 < \alpha < 0$.

Case I. For $0 < \alpha < 1$, we have from the definitions of
 $\frac{x^{-1}}{d^{\alpha}_{\eta,\alpha}}$ and $\frac{f}{e^{\alpha}_{\eta,\alpha}}$ that

$$\begin{aligned} Pf(x) &\equiv \frac{x^{-1}}{d^{\alpha}_{\eta,\alpha}} \frac{f}{e^{\alpha}_{\eta,\alpha}} f(x) = \frac{x}{d^{\alpha}_{\eta+\alpha,-\alpha}} \frac{f}{e^{\alpha}_{\eta,\alpha}} f(x) \\ &= \frac{x^{-2\eta-1}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_d^x (x^2-y^2)^{-\alpha} y^{2\eta+2\alpha+1} \frac{2y^{-2\alpha-2\eta}}{\Gamma(\alpha)} dy \int_e^f (y^2-u^2)^{\alpha-1} \\ &\quad \times u^{2\eta+1} f(u) du \end{aligned}$$

If $f \leq d$, then this integral is real. Changing the order of
integration, we get

$$(4.15) \quad Pf(x) = \frac{2x^{-2\eta-1}}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dx} \int_e^f u^{2\eta+1} f(u) du \int_d^x y(x^2-y^2)^{-\alpha} (y^2-u^2)^{\alpha-1} dy$$

In the inner integral, we put

$$(4.16) \quad y^2 = x^2 \cos^2 \theta + d^2 \sin^2 \theta$$

and use equation

$$(4.17) \quad {}_2F_1(a,b;c;Z) = \frac{2\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^{\pi/2} \frac{(\sin \theta)^{2b-1} (\cos \theta)^{2c-2b-1}}{(1-Z \sin^2 \theta)^a} d\theta$$

of Erdelyi [28], then the inner integral in (4.15) is equal to

$$(4.18) \quad \frac{1}{2} (1-\alpha)^{-1} Z^{1-\alpha} {}_2F_1(1-\alpha, 1-\alpha; 2-\alpha; Z)$$

where

$$Z = \frac{x^2 - d^2}{x^2 - u^2}, \quad \frac{dZ}{dx} = \frac{2x(d^2 - u^2)}{(x^2 - u^2)^2}.$$

Substituting (4.18) in (4.15)

$$\begin{aligned} Pf(x) &= \frac{\sin \pi \alpha}{\pi(1-\alpha)} x^{-2\eta-1} \int_e^f u^{2\eta+1} f(u) \frac{d}{dZ} [Z^{1-\alpha} {}_2F_1(1-\alpha, 1-\alpha; 2-\alpha; Z)] \frac{2x(d^2 - u^2)}{(x^2 - u^2)^2} du \\ &= \frac{2 \sin \pi \alpha}{\pi} x^{-2\eta} \int_e^f \frac{u^{2\eta+1} (d^2 - u^2) f(u)}{(x^2 - u^2)^2} Z^{-\alpha} {}_2F_1(1-\alpha, 1-\alpha; 1-\alpha; Z) du \end{aligned}$$

Since

$$(4.19) \quad {}_2F_1(1-\alpha, 1-\alpha; 1-\alpha; Z) = (1-Z)^{\alpha-1}$$

$$\therefore Pf(x) = \frac{2 \sin \pi \alpha}{\pi} x^{-2\eta} (x^2 - d^2)^{-\alpha} \int_e^f \frac{u^{2\eta+1} (d^2 - u^2) f(u) du}{x^2 - u^2}$$

Case II. When $-1 < \alpha < 0$

$$\begin{aligned}
Pf(x) &= d^{\eta, \alpha}_{\eta, \alpha}^{-1} f_{\eta, \alpha} f(x) = d^{\eta, \alpha}_{\eta + \alpha, -\alpha} f_{\eta, \alpha} f(x) \\
&= \frac{2x^{-2\eta}}{\Gamma(-\alpha)} \int_d^x (x^2 - y^2)^{-\alpha-1} y^{2\eta+2\alpha+1} \frac{y^{-2\eta-2\alpha-1}}{\Gamma(1+\alpha)} dy \frac{d}{dy} \int_e^f (y^2 - u^2)^\alpha u^{2\eta+1} f(u) du.
\end{aligned}$$

Differentiating the inner integral under the integral sign, we have

$$Pf(x) = \frac{2x^{-2\eta}}{\Gamma(-\alpha)} \int_d^x (x^2 - y^2)^{-\alpha-1} \frac{2y}{\Gamma(\alpha)} dy \int_e^f (y^2 - u^2)^{\alpha-1} u^{2\eta+1} f(u) du$$

Changing the order of integration, we obtain

$$\begin{aligned}
(4.20) \quad Pf(x) &= \frac{2x^{-2\eta}}{\Gamma(\alpha)\Gamma(-\alpha)} \int_e^f u^{2\eta+1} f(u) du \int_d^x 2y(x^2 - y^2)^{-\alpha-1} (y^2 - u^2)^{\alpha-1} dy \\
&= \frac{2x^{-2\eta}}{\Gamma(\alpha)\Gamma(-\alpha)} \int_e^f u^{2\eta+1} f(u) du \int_d^x \left(\frac{x^2 - y^2}{y^2 - u^2}\right)^{-\alpha-1} (u^2 - x^2)^{-1} d\left(\frac{x^2 - y^2}{y^2 - u^2}\right) \\
&= \frac{2x^{-2\eta}}{(-\alpha)\Gamma(\alpha)\Gamma(-\alpha)} \int_e^f u^{2\eta+1} f(u) (x^2 - d^2)^{-\alpha} \frac{(d^2 - u^2)^\alpha}{x^2 - u^2} du \\
&= \frac{2 \sin \pi\alpha}{\pi} x^{-2\eta} (x^2 - d^2)^{-\alpha} \int_e^f \frac{u^{2\eta+1} (d^2 - u^2)^\alpha f(u) du}{x^2 - u^2}.
\end{aligned}$$

Hence the lemma is completely proved.

Lemma 4.2. Let $f_{\eta, \alpha}^K$, $d_{\eta, \alpha}^{K-1}$ be operators as given in (4.4),

(4.5) and (4.12). Then

$$(4.21) \quad \frac{d_{K_{\eta,\alpha}}^{-1}}{x^{\eta,\alpha}} f_{K_{\eta,\alpha}}(x) = \frac{2 \sin \pi \alpha}{\pi} x^{2\eta+2\alpha} (d^2 - x^2)^{-\alpha} \int_e^f \frac{(t^2 - d^2)^\alpha t^{-2\alpha-2\eta+1} f(t)}{t^2 - x^2} dt$$

provided $x < d \leq e < f$.

The proof of this lemma is similar to that of Lemma 4.1.

2. Extensions. Let us consider quadruple-integral equations

$$(4.22a) \quad \int_0^\infty \xi^{-\gamma} \psi(\xi) J_\nu(\xi x) d\xi = f(x) \quad 0 < x < a$$

$$(4.22b) \quad \int_0^\infty \psi(\xi) J_\nu(\xi x) d\xi = g(x) \quad a < x < b$$

$$(4.22c) \quad \int_0^\infty \xi^{-\gamma} \psi(\xi) J_\nu(\xi x) d\xi = h(x) \quad b < x < c$$

$$(4.22d) \quad \int_0^\infty \psi(\xi) J_\nu(\xi x) d\xi = k(x) \quad c < x < \infty$$

We determine a function $\phi(\xi)$ which is such that

$$(4.23a) \quad \int_0^\infty \phi(\xi) J_\nu(\xi x) d\xi = s(x) \quad 0 < x < a$$

$$(4.23b) \quad \int_0^{\infty} \phi(\xi) J_{\nu}(\xi x) d\xi = g(x) \quad a < x < b$$

$$(4.23c) \quad \quad \quad = m(x) \quad b < x < c$$

$$(4.23d) \quad \quad \quad = k(x) \quad c < x < \infty$$

where $s(x)$ and $m(x)$ are any convenient functions of x .

$\phi(\xi)$ can be determined by usual Hankel inversion theorem and so we can find a function $r(x)$ which is such that

$$(4.24) \quad \int_0^{\infty} \xi^{-\gamma} \phi(\xi) J_{\nu}(\xi x) d\xi = r(x)$$

Substitute $\chi(\xi) = \psi(\xi) - \phi(\xi)$ and subtract equations (4.24), (4.23b), (4.24) and (4.23d) from (4.22a), (4.22b), (4.22c) and (4.22d) respectively and we have

$$(4.25a) \quad \int_0^{\infty} \xi^{-\gamma} \chi(\xi) J_{\nu}(\xi x) d\xi = f(x) - r(x) \quad 0 < x < a$$

$$(4.25b) \quad \int_0^{\infty} \chi(\xi) J_{\nu}(\xi x) d\xi = 0 \quad a < x < b$$

$$(4.25c) \quad \int_0^{\infty} \xi^{-\gamma} \chi(\xi) J_{\nu}(\xi x) d\xi = h(x) - r(x) \quad b < x < c$$

$$(4.25d) \quad \int_0^{\infty} \chi(\xi) J_{\nu}(\xi x) d\xi = 0 \quad c < x < \infty$$

These equations can be solved by the method given in the next section and thus $\chi(\xi)$ is determined. Hence $\psi(\xi)$ is found.

3. Solution of the equations (4.1). We transform the equations (4.1) into a form to which the operational theory is applicable by substituting

$$(4.26) \quad \psi(\xi) = \xi \phi(\xi); \quad f(\rho) = (2/\rho)^{2\alpha} F(\rho);$$

$$g(\rho) = (2/\rho)^2 G(\rho)$$

by means of which we get

$$(4.27a) \quad L_1(\alpha, \rho) \equiv 2^{2\alpha} \rho^{-2\alpha} \int_0^{\infty} \xi^{1-2\alpha} \phi(\xi) J_{\nu}(\rho \xi) d\xi = f_1(\rho) \quad 0 < \rho < a$$

$$(4.27b) \quad L_2(\beta, \rho) \equiv 2^{2\beta} \rho^{-2\beta} \int_0^{\infty} \xi^{1-2\beta} \phi(\xi) J_{\nu}(\rho \xi) d\xi = g_2(\rho) \quad a < \rho < b$$

$$(4.27c) \quad L_3(\alpha, \rho) \equiv 2^{2\alpha} \rho^{-2\alpha} \int_0^{\infty} \xi^{1-2\alpha} \phi(\xi) J_{\nu}(\rho \xi) d\xi = f_3(\rho) \quad b < \rho < c$$

$$(4.27d) \quad L_4(\beta, \rho) \equiv 2^{2\beta} \rho^{-2\beta} \int_0^{\infty} \xi^{1-2\beta} \phi(\xi) J_{\nu}(\rho \xi) d\xi = g_4(\rho) \quad \rho > c$$

Let I_1 denote the interval $(0,a)$, I_2 the interval (a,b) , I_3 the interval (b,c) and I_4 the interval (c,∞) . For a function f in L_2 we shall write $f_1 + f_2 + f_3 + f_4$, where

$$f_i = f \text{ on } I_i \text{ and } = 0 \text{ on } I_j \quad i, j = 1, 2, 3, 4; i \neq j$$

Using S -operator defined in (4.6), the integral equations (4.27) reduce to the form

$$(4.28) \quad S_{\frac{1}{2}\nu-\alpha, 2\alpha} \phi(\rho) = f(\rho)$$

$$(4.29) \quad S_{\frac{1}{2}\nu-\beta, 2\beta} \phi(\rho) = g(\rho).$$

Here f_1, g_2, f_3 and g_4 are prescribed but g_1, f_2, g_3 and f_4 are to be determined. We consider the case in which $G_2 \equiv 0 \equiv G_4$ and take as trial solution

$$(4.30) \quad \phi(\rho) \equiv S_{\frac{1}{2}\nu+\beta, -\alpha-\beta}^{\ell}(\rho).$$

Substituting this value of $\phi(\rho)$ in (4.28), (4.29) and using formulas (4.9), (4.10), we have

$$(4.31) \quad f = I_{\frac{1}{2}\nu+\beta, \alpha-\beta}^{\ell}$$

$$(4.32) \quad g = K_{\frac{1}{2}\nu-\beta, \beta-\alpha} \ell.$$

Also, we have

$$(4.33) \quad \ell = I_{\frac{1}{2}\nu+\beta, \alpha-\beta}^{-1} f,$$

$$(4.34) \quad \ell = K_{\frac{1}{2}\nu-\beta, \beta-\alpha}^{-1} g.$$

We proceed to determine ℓ and the subscripts on all the operators will be dropped for brevity sake. All I 's will be supposed to have subscripts $\frac{1}{2}\nu+\beta, \alpha-\beta$ understood and all K 's to have $\frac{1}{2}\nu-\beta, \beta-\alpha$.

Evaluate (4.33) on I_1 , then

$$(4.35) \quad \ell_1 = {}^\rho I^{-1} f_1, \text{ i.e., } \ell_1 \text{ is determined.}$$

Taking (4.34) on I_4 we have

$$(4.36) \quad \ell_4 = {}^\rho K^{-1} g_4 = {}^\rho K^{-1} 0 = 0.$$

Evaluating (4.32) on I_2 we get

$$g_2 = {}^b_\rho K \ell_2 + {}^c_b K \ell_3 + {}^\infty_c K \ell_4$$

i.e.
$$0 = \frac{b}{\rho} K \ell_2 + \frac{c}{b} K \ell_3 + \frac{\infty}{c} K 0$$

which gives

$$(4.37) \quad \ell_2 = - \frac{b}{\rho} K^{-1} \frac{c}{b} K \ell_3$$

where $\rho < b \leq b < c$. Applying Lemma 4.2 to (4.37) we obtain

$$(4.38) \quad \ell_2(\rho) = - \frac{2 \sin \pi(\beta-\alpha)}{\pi} \rho^{\nu-2\alpha} (b^2-\rho^2)^{\alpha-\beta} \int_b^c \frac{(t^2-b^2)^{\beta-\alpha} t^{-\nu+2\alpha+1} \ell_3(t)}{t^2-\rho^2} dt$$

Finally evaluating (4.31) on I_3 , we have

$$f_3 = \frac{a}{o} I \ell_1 + \frac{b}{a} \ell_2 + \frac{\rho}{b} I \ell_3.$$

Simplifying we get

$$(4.39) \quad \ell_3 = \frac{\rho}{b} I^{-1} f_3 - \frac{\rho}{b} I^{-1} \frac{a}{o} I \ell_1 - \frac{\rho}{b} I^{-1} \frac{b}{a} I \ell_2.$$

Since f_3 and ℓ_1 are known functions. The function

$$(4.40) \quad d(\rho) = \frac{\rho}{b} I^{-1} f_3(\rho) - \frac{\rho}{b} I^{-1} \frac{a}{o} I \ell_1(\rho)$$

is known. Substituting (4.40) in (4.39) we have

$$(4.41) \quad \ell_3(\rho) = d(\rho) - \frac{\rho}{b} I^{-1} \frac{b}{a} I \ell_2(\rho)$$

Applying Lemma 4.1 to the second term on the right hand side of (4.41) and substituting the value of $\ell_2(\rho)$ from (4.38) we obtain

$$\begin{aligned} \ell_3(\rho) = d(\rho) + \frac{2 \sin \pi(\alpha-\beta)}{\pi} \rho^{-\nu-2\beta} (\rho^2 - b^2)^{\beta-\alpha} \\ \int_a^b (b^2 - y^2)^{\alpha-\beta} y^{\nu+2\beta+1} \left[\frac{2 \sin \pi(\beta-\alpha)}{\pi} y^{\nu-2\alpha} (b^2 - y^2)^{\alpha-\beta} \right. \\ \left. \int_b^c \frac{(t^2 - b^2)^{\beta-\alpha} t^{-\nu+2\alpha+1} \ell_3(t)}{t^2 - y^2} dt \right] \frac{1}{\rho^2 - y^2} dy. \end{aligned}$$

Inverting the order of integration, we get

$$\begin{aligned} (4.42) \quad \ell_3(\rho) = d(\rho) - \frac{4 \sin^2 \pi(\alpha-\beta)}{\pi^2} \int_b^c \{ \rho^{-\nu-2\beta} (\rho^2 - b^2)^{\beta-\alpha} \\ (t^2 - b^2)^{\beta-\alpha} t^{-\nu+2\alpha+1} \int_a^b (b^2 - y^2)^{2(\alpha-\beta)} y^{2\nu-2\alpha+2\beta+1} \\ \frac{1}{(t^2 - y^2)(\rho^2 - y^2)} dy \} \ell_3(t) dt. \end{aligned}$$

Putting

$$\frac{4 \sin^2 \pi(\alpha-\beta)}{\pi^2} = \lambda$$

and expression within the curly brackets in (4.42) = $-K(\rho, t)$, we obtain

$$(4.43) \quad \ell_3(\rho) = d(\rho) + \lambda \int_b^c K(\rho, t) \ell_3(t) dt$$

which is a Fredholm's integral equation of the second kind and can be solved by known methods. The equations (4.35), (4.36), (4.38), and (4.43) completely determine ℓ and hence our problem is solved.

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